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Strong and Plancherel–Rotach Asymptotics of Non-diagonal Laguerre–Sobolev Orthogonal Polynomials

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We study properties of the monic polynomials $\{Q_n\}_{n\in\mathbb{N}}$ orthogonal with respect to the Sobolev inner product

$$(p, q)_S = \int_0^\infty (p, p') \begin{pmatrix} 1 & \mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} x^\alpha e^{-x}\, dx,$$

where $\lambda - \mu^2 > 0$ and $\alpha > -1$. This inner product can be expressed as

$$(p, q)_S = \int_0^\infty p(x)\, q(x)((\mu+1)\, x - \alpha\mu)\, x^{\alpha-1} e^{-x}\, dx + \lambda \int_0^\infty p'q'x^\alpha e^{-x}\, dx,$$

when $\alpha > 0$. In this way, the measure which appears in the first integral is not positive on $[0, \infty)$ for $\mu \in \mathbb{R} \setminus [-1, 0]$. The aim of this paper is the study of analytic properties of the polynomials Q_n. First we give an explicit representation for Q_n using an algebraic relation between Sobolev and Laguerre polynomials together with a recursive relation for $\tilde{k}_n = (Q_n, Q_n)_S$. Then we consider analytic aspects. We first establish the strong asymptotics of Q_n on $\mathbb{C} \setminus [0, \infty)$ when $\mu \in \mathbb{R}$ and we also obtain an asymptotic expression on the oscillatory region, that is, on $(0, \infty)$. Then we study the Plancherel–Rotach asymptotics for the Sobolev polynomials $Q_n(nx)$ on $\mathbb{C} \setminus [0, 4]$ when $\mu \in (-1, 0]$. As a consequence of these results we obtain the accumulation sets of zeros and of the scaled zeros of Q_n. We also give a Mehler–Heine type formula for the Sobolev polynomials which is valid on compact subsets

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of $\mathbb{C}$ when $\mu \in (-1, 0]$, and hence in this situation we obtain a more precise result about the asymptotic behaviour of the small zeros of Q_n. This result is illustrated with three numerical examples.

Key Words: Sobolev orthogonal polynomials; Laguerre polynomials; Bessel functions; scaled polynomials; asymptotics; Plancherel–Rotach asymptotics.

1. INTRODUCTION

The study of the asymptotic properties of the polynomials orthogonal with respect to a Sobolev inner product

$$(p, q)_S = \int_{\mathbb{R}} pq \, d\mu_0 + \int_{\mathbb{R}} p'q' \, d\mu_1, \tag{1}$$

where μ_0 and μ_1 are measures supported on sets with infinitely many points in $\mathbb{R}$ and such that their absolutely continuous components do not vanish (such orthogonal polynomials are called continuous Sobolev orthogonal polynomials) has known an increasing development in the last few years. Probably, one of the first papers in this direction was [5]. A survey on this topic is given in the recent paper [6].

A continuous non-diagonal Sobolev inner product

$$(p, q)_S = \int_a^b (p, p', ..., p^{(k)}) \, W(x) \begin{pmatrix} q \\ q' \\ \vdots \\ q^{(k)} \end{pmatrix} dx, \tag{2}$$

where W is a positive definite matrix function, integrable in some interval $[a, b] \subset \mathbb{R}$ was analyzed by Schäfke and Wolf in [12] taking into consideration the analog of the classical orthogonal polynomials (Laguerre, Hermite and Jacobi).

A particular situation of (2) is when $k=1$ and $W(x) = \left(\begin{smallmatrix} 1 & \mu \\ \mu & \lambda \end{smallmatrix}\right) w(x)$, where w is a weight function satisfying a Pearson equation. The Jacobi case was treated in [9] from the point of view of the analytic properties of the corresponding sequences of orthogonal polynomials. Their asymptotic behavior in $\overline{\mathbb{C}} \backslash [-1, 1]$ as well as the distribution of their zeros were studied.

The present contribution deals with the study of asymptotic properties of polynomials orthogonal with respect to the inner product

$$(p, q)_S = \int_0^\infty (p, p') \begin{pmatrix} 1 & \mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} q \\ q' \end{pmatrix} x^\alpha e^{-x} \, dx, \tag{3}$$

with $\alpha > -1$. We assume $\lambda - \mu^2 > 0$. This ensures that $(\cdot, \cdot)_S$ is a positive definite inner product and the existence of a unique sequence $\{Q_n\}$ of monic polynomials orthogonal with respect to (3).

Notice that for $\alpha > 0$, integration by parts yields

$$(p, q)_S = \int_0^\infty p(x)\, q(x) [(\mu+1)\, x - \alpha\mu]\, x^{\alpha-1} e^{-x}\, dx + \lambda \int_0^\infty p'(x)\, q'(x)\, x^\alpha e^{-x}\, dx. \tag{4}$$

If we write $s(x) = (\mu+1)\, x - \alpha\mu$, then

$$(p, q)_S = \int_0^\infty p(x)\, q(x)\, s(x)\, x^{\alpha-1} e^{-x}\, dx + \lambda \int_0^\infty p'(x)\, q'(x)\, x^\alpha e^{-x}\, dx,$$

and $s(x)$ is a polynomial of degree at most one. Actually, if $\mu \neq -1$, the degree of $s(x)$ is exactly one and if $\mu = -1$ we have the trivial case as it will be shown later.

Thus, if $\mu \in \mathbb{R} \setminus [-1, 0]$ the inner product is an example of a Sobolev inner product where the measure acting on the standard part changes sign in its support. In some sense, the integral term involving derivatives guarantees the positive definiteness of (4). If $\mu \in (-1, 0]$ the inner product is an example of a Sobolev inner product associated with a coherent pair of measures in the unbounded case (see [7] for the classification of coherent pairs). Moreover we also recover some examples of coherent pairs as particular cases of our inner product.

Denote by $L_n^{(\alpha)}(x)$ the nth monic Laguerre polynomial. We know that $L_n^{(\alpha)}(x)$ are polynomials orthogonal with respect to

$$\langle p, q \rangle = \int_0^\infty p(x)\, q(x)\, x^\alpha e^{-x}\, dx, \qquad \alpha > -1.$$

For $\alpha \in \mathbb{R}$ we know the explicit representation of such polynomials (see [11, p. 201] or [13, p. 102]):

$$L_n^{(\alpha)}(x) = (-1)^n\, n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}. \tag{5}$$

The following theorem summarizes some of the properties of Laguerre polynomials which will be used later (see [11, 13]):

THEOREM 1. (a) *Let* $\alpha \in \mathbb{R}$. *Then*

$$L_{n+1}^{(\alpha-1)}(x) = L_{n+1}^{(\alpha)}(x) + (n+1)\, L_n^{(\alpha)}(x),$$
$$(L_{n+1}^{(\alpha-1)}(x))' = (n+1)\, L_n^{(\alpha)}(x).$$

(b) (*Perron's formula*) *Let* $\alpha \in \mathbb{R}$. *Then*

$$\frac{(-1)^n\, L_n^{(\alpha)}(x)}{n!} = 2^{-1}\pi^{-1/2} e^{x/2} (-x)^{-\alpha/2 - 1/4}\, n^{\alpha/2 - 1/4} e^{2(-nx)^{1/2}} (1 + O(n^{-1/2})).$$

This relation holds for x *in the complex plane cut along the positive real semiaxis; both* $(-x)^{-\alpha/2-1/4}$ *and* $(-x)^{1/2}$ *must be taken real and positive if* $x < 0$. *The bound of the remainder holds uniformly in every closed domain which does not overlap the positive real semiaxis.*

(c) *Uniformly on compact subsets of* $\mathbb{C} \setminus [0, \infty)$,

$$\lim_{n \to \infty} \frac{L_{n+1}^{(\alpha)}(x)}{n L_n^{(\alpha)}(x)} = -1.$$

(d) *Let* $\alpha > -1$. *Then*

$$k_n^{(\alpha)} := \langle L_n^{\alpha)}, L_n^{(\alpha)} \rangle = n!\, \Gamma(n + \alpha + 1).$$

(e) *Recurrence relation*:

$$x L_n^{(\alpha)}(x) = L_{n+1}^{(\alpha)}(x) + (2n + \alpha + 1)\, L_n^{(\alpha)}(x) + n(n+\alpha)\, L_{n-1}^{(\alpha)}(x),$$
$$n = 0, 1, 2, \ldots$$

with $L_{-1}(x) = 0$ *and* $L_0(x) = 1$.

In what follows we suppose $\mu \neq -1$ because for $\mu = -1$ we have $Q_n(x) = L_n^{(\alpha-1)}(x)$, which is straightforward from (4) using (a) of Theorem 1 (or Lemma 1).

The main goal of this paper is to obtain analytic results for the Sobolev polynomials (Q_n) associated with the inner product (3) and to give a simple way to calculate them. First, we deduce a recurrence relation for $\tilde{k}_n$, using an important algebraic relation between Sobolev polynomials and Laguerre polynomials. Both things allow us to obtain an explicit representation of Q_n. Second, we obtain the asymptotic behavior of $\tilde{k}_n := (Q_n, Q_n)_S$. These results are necessary to face the analytic properties. We give different types of asymptotic results for Q_n. The relative asymptotics $Q_n(x)/L_n^{(\alpha-1)}(x)$ in $\mathbb{C} \setminus [0, \infty)$ as well as the strong asymptotics of $\{Q_n\}$ in $\mathbb{C} \setminus [0, \infty)$ are obtained. As a fairly direct consequence, the distribution of zeros of Q_n is given. When $\mu=0$ (the diagonal case) the strong asymptotics of Q_n on $\mathbb{C}\backslash[0,\infty)$ was obtained in [4] using tools and techniques which are specific for this particular case. We also obtain an asymptotic result for Q_n on $(0,\infty)$ for $\mu\in\mathbb{R}$.

Next, we study the Plancherel–Rotach asymptotics of the Sobolev polynomials. Scaling the variable, we can deduce the relative asymptotics for the scaled Sobolev polynomials with respect to the scaled Laguerre polynomials when $\mu\in(-1,0]$. Taking this into account as well as the Plancherel–Rotach asymptotics for the sequence $\{L_n(nx)\}$, we get the analogous result for the scaled Sobolev polynomials $\{Q_n(nx)\}$. Notice that in a recent paper [10] the study of the nth root asymptotics for the scaled Sobolev–Laguerre polynomials on the oscillatory region is presented.

Finally, we give a Mehler–Heine type formula for Q_n when $\mu\in(-1,0]$. From this result we obtain a limit relation for the zeros of Q_n in terms of the zeros of the Bessel function $J_{\alpha-1}(x)$ defined as (see, e.g., [13, p. 15]):

$$J_k(x)=\sum_{j=0}^{\infty}\frac{(-1)^j\,(x/2)^{2j+k}}{j!\,\Gamma(j+k+1)}.$$

Note that the zeros of $z^{-k}J_k(x)$ are real if $k>-1$ (see [13, p. 193]).

2. MAIN RESULTS

First, we give a recurrence relation for $\tilde{k}_n=(Q_n,Q_n)_S$,

PROPOSITION 1. *For $n\geqslant 1$,*

$$\tilde{k}_n=k_n^{(\alpha)}+(1+\lambda+2\mu)\,n^2k_{n-1}^{(\alpha)}-(1+\mu)^2\,n^2\,\frac{(k_{n-1}^{(\alpha)})^2}{\tilde{k}_{n-1}}. \tag{6}$$

Using Proposition 1 and the well-known Poincaré Theorem we get

PROPOSITION 2. *One has*

$$l:=\lim_{n\to\infty}\frac{k_n^{(\alpha)}}{\tilde{k}_n}=\frac{\lambda+2(1+\mu)-\sqrt{\lambda^2+4\lambda(1+\mu)}}{2(1+\mu)^2}<1.$$

The proposition above allows us to deduce the relative asymptotics $Q_n/L_n^{(\alpha-1)}$ as $n\to\infty$

THEOREM 2. *Uniformly on compact subsets of* $\mathbb{C}\backslash[0, \infty)$,

$$\lim_{n\to\infty} \frac{Q_n(x)}{L_n^{(\alpha-1)}(x)} = \frac{2(1+\mu)}{\sqrt{\lambda^2+4\lambda(1+\mu)}-\lambda}.$$

COROLLARY 1 (Strong asymptotics of Q_n). *Uniformly on compact subsets of* $\mathbb{C}\backslash[0, \infty)$,

$$\frac{(-1)^n Q_n(x)}{n!\, n^{\alpha/2-3/4} e^{2(-nx)^{1/2}}} = \frac{(1+\mu)}{(\sqrt{\lambda^2+4\lambda(1+\mu)}-\lambda)\sqrt{\pi}} e^{x/2}(-x)^{-\alpha/2+1/4}(1+O(n^{-1/2})).$$

It is known (see [1]) that when $\mu \in (-1, 0]$ the zeros of Q_n are all real. Moreover, if $\alpha \geqslant 0$, they are all positive and, if $\alpha \in (-1, 0)$, there is at most one negative zero. Next, we give a first result about zeros of $Q_n(x)$ with $\mu \in \mathbb{R}$.

COROLLARY 2. *The zeros of the non-diagonal Laguerre–Sobolev orthogonal polynomials* Q_n *accumulate on* $[0, \infty)$, *when* $\alpha > -1$.

Furthermore, we can also obtain an asymptotic result on $(0, \infty)$.

THEOREM 3. *Uniformly on compact subsets of* $(0, \infty)$,

$$\frac{(-1)^n Q_n(x)}{n!\, n^{\alpha/2-1/2}} = e^{x/2} x^{-\alpha/2+1/2} J_{\alpha-1}(2\sqrt{nx}) + O(n^{-1/4}),$$

where $J_{\alpha-1}(x)$ *is the Bessel function defined as*

$$J_{\alpha-1}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j+\alpha-1}}{j!\, \Gamma(j+\alpha)}.$$

We have bounds for $\tilde{k}_n$,

PROPOSITION 3. *For* $n \geqslant 1$,

$$(\lambda-\mu^2)\, n^2 k_{n-1}^{(\alpha)} + k_n^{(\alpha)} \leqslant \tilde{k}_n \leqslant k_n^{(\alpha)} + (1+\lambda+2\mu)\, n^2 k_{n-1}^{(\alpha)}, \qquad n \geqslant 1.$$

From the above Proposition we deduce a uniform bound for the ratio $k_n^{(\alpha)}/\tilde{k}_n$, that is,

$$\frac{k_n^{(\alpha)}}{\tilde{k}_n} \leqslant \mathscr{C} < 1, \qquad n \geqslant 1, \tag{7}$$

where $\mathscr{C}$ can be taken as

$$\mathscr{C} = \frac{1}{1 + \dfrac{\lambda - \mu^2}{\max\{1, 1+\alpha\}}}. \tag{8}$$

Now, we look for an asymptotic result of Plancherel–Rotach type for the scaled polynomials $Q_n(nx)$ in the exterior region. An asymptotic result of Plancherel–Rotach type in the oscillatory region is obtained in [10] when $\mu \in (-1, 0]$ (Laguerre coherent pairs of type I, see [7]).

THEOREM 4. *Uniformly on compact subsets of* $\mathbb{C} \backslash [0, 4]$,

(a) *Letting* $\mu \in (-1, 0]$ *and* $\alpha > 0$,

$$\lim_{n \to \infty} \frac{Q_n(nx)}{L_n^{(\alpha-1)}(nx)} = \frac{\varphi\left(\dfrac{x-2}{2}\right)}{\varphi\left(\dfrac{x-2}{2}\right) + l(1+\mu)}. \tag{9}$$

(b) *Letting* $\mu \in (-1, 0]$ *and* $\alpha > -1$,

$$\lim_{n \to \infty} \frac{Q_n(nx)}{L_n^{(\alpha)}(nx)} = \frac{\varphi\left(\dfrac{x-2}{2}\right) + 1}{\varphi\left(\dfrac{x-2}{2}\right) + l(1+\mu)}, \tag{10}$$

where $\varphi(x) = x + \sqrt{x^2 - 1}$ *with* $\sqrt{x^2 - 1} > 0$ *if* $x > 1$, *and* l *is given in Proposition* 2.

From this result and using the scaled asymptotics for the Laguerre polynomials, we get

COROLLARY 3. *Denote by* $x_{n,i}$, $i = 1, ..., n$ *the zeros in increasing order of* Q_n *and* $\mu \in (-1, 0]$, *then the contracted zeros of* Q_n, $(x_{n,i}/n)$ $i = 1, ..., n$, *accumulate on* $[0, 4]$, *and moreover, the asymptotic distribution of the contracted zeros has density* $(2\pi)^{-1} \sqrt{4 - x}/\sqrt{x}$.

COROLLARY 4 (Plancherel–Rotach asymptotics). *If* $\mu \in (-1, 0]$, *then uniformly on compact subsets of* $\mathbb{C} \backslash [0, 4]$,

$$\lim_{n \to \infty} \frac{2^n \sqrt{2\pi n}\, Q_n(nx)}{n!\, (x-2+\sqrt{x^2-4x})^n \exp\left(\dfrac{2nx}{x+\sqrt{x^2-4x}}\right)}$$

$$= \frac{\varphi\left(\dfrac{x-2}{2}\right)+1}{\varphi\left(\dfrac{x-2}{2}\right)+l(1+\mu)}\, 2^{-\alpha-1/2} x^{-\alpha} (x-2+\sqrt{x^2-4x})^{1/2}$$

$$\times (x+\sqrt{x^2-4x})^{\alpha} (\sqrt{x^2-4x})^{-1/2},$$

taking into account that the square roots in the above formula are negative if x *is negative.*

Next, we deduce a Mehler–Heine type formula for Q_n. We consider the following modification of the Bessel function

$$x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}) = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!\, \Gamma(j+\alpha)}.$$

Then we can establish the asymptotics near 0, complementing Theorem 3:

THEOREM 5. *Let* $\mu \in (-1, 0]$. *Then, for* $\alpha > -1$,

$$\lim_{n \to \infty} \frac{(-1)^n Q_n(x/n)}{n!\, n^{\alpha-1}} = \frac{2(1+\mu)}{\sqrt{\lambda^2+4\lambda(1+\mu)}-\lambda}\, x^{-(\alpha-1)/2} J_{\alpha-1}(2\sqrt{x}) \tag{11}$$

holds uniformly for x *on compact subsets of* $\mathbb{C}$.

From this Theorem we can obtain additional information about of zeros of Q_n when $\mu \in (-1, 0]$. By Corollary 2, we know that these zeros accumulate in $[0, \infty)$ when $n \to \infty$. Now, by (11) the small zeros of Q_n satisfy the following asymptotic property:

PROPOSITION 4. *If* $\mu \in (-1, 0]$, $\alpha > -1$ *and* $x_{n,i}$ *are the zeros of* Q_n, *then*

$$\lim_{n \to \infty} n x_{n,i} = \frac{j_{\alpha-1,i}^2}{4},$$

where, if all the zeros of Q_n *are non-negative,* $j_{\alpha-1,i}$ *is ith positive real zero of Bessel function* $J_{\alpha-1}(x)$ *and, if* Q_n *has one negative zero, then* $j_{\alpha-1,1}$ *is* *any of the two complex zeros of* $J_{\alpha-1}(x)$ *and, for* $i \geqslant 2$, $j_{\alpha-1,i}$ *is* $(i-1)$*th positive real zero of* $J_{\alpha-1}(x)$.

Remark 1. When $\alpha \in (-1, 0)$, Q_n has at most one negative zero. In this situation $J_{\alpha-1}(x)$ has exactly two complex zeros on the imaginary axis, thus the limit value $j^2_{\alpha-1,1}/4$ is negative.

We illustrate this result with three numerical examples. In these examples, we compute $nx_{n,i}$ for $i=1, ..., 5$ and $n=25, 50, 100, 200, 300$ and compare with the limit values $j^2_{\alpha-1,i}/4$.

EXAMPLE 1. $\lambda=4.5$, $\alpha=1$ and $\mu=-0.5$.

	1st Zero	2nd Zero	3rd Zero	4th Zero	5th Zero
$n=25$	1.423597691	7.506777687	18.475193396	34.373457650	55.263999713
$n=50$	1.434478208	7.559703167	18.585675024	34.525537630	55.395117910
$n=100$	1.440082185	7.588093425	18.650413696	34.632202794	55.537671207
$n=200$	1.442925484	7.602785665	18.685243739	34.693441211	55.628662289
$n=300$	1.443879401	7.607757879	18.697225161	34.715047218	55.661963418
$\frac{j^2_{\alpha-1,i}}{4}$	**1.445796491**	**7.617815586**	**18.721751699**	**34.760071107**	**55.733075905**

EXAMPLE 2. $\lambda=4.5$, $\alpha=1$ and $\mu=0$ (diagonal case).

	1st Zero	2nd Zero	3rd Zero	4th Zero	5th Zero
$n=25$	1.428828896	7.534309400	18.542719079	34.498466787	55.463683324
$n=50$	1.437030000	7.573145181	18.618696247	34.586808979	55.493278627
$n=100$	1.441343336	7.594737989	18.666741939	34.662514573	55.586262953
$n=200$	1.443552511	7.606089388	18.693362868	34.708515216	55.652830333
$n=300$	1.444296644	7.609956291	18.702627983	34.725078312	55.678046603
$\frac{j^2_{\alpha-1,i}}{4}$	**1.445796491**	**7.617815586**	**18.721751699**	**34.760071107**	**55.733075905**

EXAMPLE 3. $\lambda=5$, $\alpha=-0.9$ and $\mu=-0.25$.

	1st Zero	2nd Zero	3rd Zero	4th Zero	5th Zero
$n=25$	-0.331304673	0.615975720	7.317078637	18.994018805	35.800381920
$n=50$	-0.327321935	0.608512493	7.223516960	18.728907791	35.240186072
$n=100$	-0.325385645	0.604898564	7.179421436	18.609185233	35.000343889
$n=200$	-0.324430710	0.603119801	7.158014537	18.552368155	34.889893762
$n=300$	-0.324114318	0.602530977	7.150971819	18.533869676	34.854444681
$\frac{j^2_{\alpha-1,i}}{4}$	**−0.323484380**	**0.601359401**	**7.137023996**	**18.497524649**	**34.785568869**

Remark 2. Since we have uniform convergence in the asymptotic results obtained for Q_n, we also get asymptotic results for the derivatives of Q_n. In particular, taking derivatives in Theorem 5 we have for $\alpha > -1$ and $\mu \in (-1, 0]$,

$$\lim_{n \to \infty} \frac{(-1)^{n-1} Q_n'(x/n)}{n!\, n^{\alpha}} = \frac{2(1+\mu)}{\sqrt{\lambda^2 + 4\lambda(1+\mu)} - \lambda} x^{-\alpha/2} J_{\alpha}(2\sqrt{x}),$$

uniformly on compact subsets of $\mathbb{C}$. Thus, we have asymptotic information about the critical points $\tilde{x}_{n,i}$ of Q_n, that is,

$$\lim_{n \to \infty} n\tilde{x}_{n,i} = \frac{j_{\alpha,i}^2}{4},$$

where $j_{\alpha,i}$ has the same meaning as in Proposition 4. In the relevant paper [2] one can find information about the asymptotic distribution of zeros and critical points of Sobolev orthogonal polynomials in the bounded case.

3. PROOFS

We first obtain the following algebraic relation between Sobolev and Laguerre polynomials:

Lemma 1.

$$L_{n+1}^{(\alpha-1)}(x) = Q_{n+1}(x) + (1+\mu)(n+1)\frac{k_n^{(\alpha)}}{\tilde{k}_n} Q_n(x), \qquad n \geqslant 0. \tag{12}$$

Proof. Expanding $L_{n+1}^{(\alpha-1)}(x)$ in the basis $\{Q_j\}_{j=0}^{n+1}$ of $\mathbb{P}_{n+1}$, where $\mathbb{P}_{n+1}$ is the linear space of polynomials with degree at most $n+1$, we get

$$L_{n+1}^{(\alpha-1)}(x) = Q_{n+1}(x) + \sum_{i=0}^{n} a_i^{(n+1)} Q_i(x),$$

where

$$a_i^{(n+1)} = \frac{(L_{n+1}^{(\alpha-1)}, Q_i)_S}{\tilde{k}_i} = \frac{(L_{n+1}^{(\alpha)} + (n+1)\, L_n^{(\alpha)}, Q_i)_S}{\tilde{k}_i}.$$

Therefore, using Theorem 1(a), we obtain

$$a_i^{(n+1)}=0, \qquad i=0, ..., n-1,$$

$$a_n^{(n+1)}=(1+\mu)(n+1)\,\frac{k_n^{(\alpha)}}{\tilde{k}_n}. \quad \blacksquare$$

Proof of Proposition 1. By definition of $\tilde{k}_n$,

$$\begin{aligned}\tilde{k}_n &= (Q_n, Q_n)_S=(Q_n, L_n^{(\alpha-1)})_S\\ &= \langle Q_n, L_n^{(\alpha-1)}\rangle+\lambda\,\langle Q_n', (L_n^{(\alpha-1)})'\rangle\\ &\quad +\mu[\langle Q_n', L_n^{(\alpha-1)}\rangle+\langle Q_n, (L_n^{(\alpha-1)})'\rangle].\end{aligned}$$

Then, using Theorem 1(a), we get

$$\tilde{k}_n=k_n^{(\alpha)}+(\lambda+\mu)\,n^2k_{n-1}^{(\alpha)}+(1+\mu)\,n\,\langle Q_n, L_{n-1}^{(\alpha)}\rangle. \tag{13}$$

On the other hand, using Lemma 1 and again Theorem 1(a), we get

$$\begin{aligned}\langle Q_n, L_{n-1}^{(\alpha)}\rangle &= \left\langle L_n^{(\alpha-1)}-(1+\mu)\,n\,\frac{k_{n-1}^{(\alpha)}}{\tilde{k}_{n-1}}\,Q_{n-1}, L_{n-1}^{(\alpha)}\right\rangle\\ &= nk_{n-1}^{(\alpha)}-(1+\mu)\,n\,\frac{(k_{n-1}^{(\alpha)})^2}{\tilde{k}_{n-1}}\end{aligned} \tag{14}$$

Substituting (14) in (13), (6) follows. ∎

Relation (12) and the recurrence relation for $\tilde{k}_n=(Q_n, Q_n)_S$ that we have just proved are very useful tools to compute the polynomials $Q_n(x)$. We can express $Q_{n+1}(x)$ as a linear combination of $L_{n+1}^{(\alpha-1)}(x)$ and $Q_n(x)$ in the following way:

$$Q_{n+1}(x)=L_{n+1}^{(\alpha-1)}(x)-(1+\mu)(n+1)\,\frac{k_n^{(\alpha)}}{\tilde{k}_n}\,Q_n(x). \tag{15}$$

Since $Q_0(x)=L_0^{(\alpha-1)}(x)=1$, $k_0=\tilde{k}_0=\Gamma(\alpha+1)$ and using (6) to compute $\tilde{k}_n$, then it is easy to observe that the computation of $Q_n(x)$, using (15) in a recursive way, only needs Laguerre polynomials, the square of their norms $k_n^{(\alpha)}$ and the parameters λ and μ. We summarize this in the following result:

Corollary 5. *For* $n\geqslant 0$,

(a) $$Q_n(x)=\sum_{j=0}^{n}(-1)^j\,b_j^{(n)}\,L_{n-j}^{(\alpha-1)}(x),$$

where

$$b_0^{(n)} = 1,$$
$$b_j^{(n)} = (1+\mu)^j \prod_{i=1}^{j} (n-i+1) \frac{k_{n-i}^{(\alpha)}}{\tilde{k}_{n-i}}, \qquad j = 1, ..., n. \tag{16}$$

(b)
$$Q_n(x) = \sum_{j=0}^{n} c_j^{(n)} x^{n-j},$$

where

$$c_0^{(n)} = 1$$
$$c_j^{(n)} = \frac{(-1)^j}{(n-j)!} \sum_{i=0}^{j} \binom{n-i+\alpha-1}{j-i} (n-i)!\, b_i^{(n)}, \qquad j = 1, ..., n,$$

where $b_i^{(n)}$ *is given by* (16).

Proof. (a) It is straightforward applying (15) in a recursive way.

(b) Follows by substituting in (a) the explicit representation of $L_{n-j}^{(\alpha-1)}$ given by (5). ∎

Proof of Proposition 2. If we divide (6) by $k_n^{(\alpha)}$ and use their explicit values, we get

$$\frac{\tilde{k}_n}{k_n^{(\alpha)}} = 1 + (1+\lambda+2\mu) \frac{n}{n+\alpha} - (1+\mu)^2 \frac{n}{n+\alpha} \frac{k_{n-1}^{(\alpha)}}{\tilde{k}_{n-1}}. \tag{17}$$

We can define $s_{n+1} = (\tilde{k}_n / k_n^{(\alpha)})\, s_n$ with the initial condition $s_0 = 1$. Therefore, (17) can be rewritten as

$$s_{n+1} - \left(1 + (1+\lambda+2\mu) \frac{n}{n+\alpha}\right) s_n + (1+\mu)^2 \frac{n}{n+\alpha} s_{n-1} = 0,$$

where $s_0 = 1$ and $s_1 = \tilde{k}_1 / k_1^{(\alpha)}$. Since

$$\lim_{n\to\infty} \left(1 + (1+\lambda+2\mu) \frac{n}{n+\alpha}\right) = 2(1+\mu) + \lambda,$$

$$\lim_{n\to\infty} (1+\mu)^2 \frac{n}{n+\alpha} = (1+\mu)^2,$$

and $(2(1+\mu)^2+\lambda)^2 - 4(1+\mu)^2 = \lambda^2 + 4\lambda(1+\mu) > 0$, the roots l_1, l_2 of the characteristic equation

$$t^2 - (2(1+\mu)+\lambda)\, t + (1+\mu)^2 = 0,$$

are real, simple and positive. More precisely,

$$l_1 = \frac{\lambda + 2(1+\mu) + \sqrt{\lambda^2 + 4\lambda(1+\mu)}}{2},$$

$$l_2 = \frac{\lambda + 2(1+\mu) - \sqrt{\lambda^2 + 4\lambda(1+\mu)}}{2}.$$

Thus, by Poincaré's Theorem $\tilde{k}_n / k_n^{(\alpha)} = s_{n+1}/s_n$ converges to one of these roots. Since $\tilde{k}_n \geqslant k_n^{(\alpha)}$, we need to choose l_1

$$\lim_{n\to\infty} \frac{\tilde{k}_n}{k_n^{(\alpha)}} = l_1 > 1.$$

Therefore,

$$l := \lim_{n\to\infty} \frac{k_n^{(\alpha)}}{\tilde{k}_n} = \frac{1}{l_1} = \frac{\lambda + 2(1+\mu) - \sqrt{\lambda^2 + 4\lambda(1+\mu)}}{2(1+\mu)^2} < 1. \quad \blacksquare$$

We are now ready to prove the relative asymptotics $Q_n / L_n^{(\alpha-1)}$ as $n \to \infty$.

Proof of Theorem 2. Denote $Y_n(x) = Q_n(x)/L_n^{(\alpha-1)}(x)$. From Lemma 1 we have

$$1 = Y_{n+1}(x) + (1+\mu)\,\frac{k_n^{(\alpha)}}{\tilde{k}_n}\,(n+1)\,\frac{L_n^{(\alpha-1)}(x)}{L_{n+1}^{(\alpha-1)}(x)}\,Y_n(x). \tag{18}$$

On the other hand, some simple computations yield

$$|l(1+\mu)| < 1. \tag{19}$$

Using Theorem 1(c) and Proposition 2 we obtain

$$\lim_{n\to\infty} \left| (1+\mu)\,\frac{k_n^{(\alpha)}}{\tilde{k}_n}\,(n+1)\,\frac{L_n^{(\alpha-1)}(x)}{L_{n+1}^{(\alpha-1)}(x)} \right| = |l(1+\mu)| < 1,$$

locally uniformly on $\mathbb{C} \backslash [0, \infty)$.

Thus, for a fixed compact set $K \subset \mathbb{C} \backslash [0, \infty)$ there exist constants s with $0 < s < 1$ and $n_0 \in \mathbb{N}$ such that when $x \in K$ and $n \geqslant n_0$,

$$\left| (1+\mu)\,\frac{k_n^{(\alpha)}}{\tilde{k}_n}\,(n+1)\,\frac{L_n^{(\alpha-1)}(x)}{L_{n+1}^{(\alpha-1)}(x)} \right| \leqslant s.$$

Therefore,

$$|Y_{n+1}(x)| \leqslant 1 + s\,|Y_n(x)|, \qquad n \geqslant n_0, \qquad x \in K.$$

Hence, $\{Y_n\}$ is uniformly bounded on compact subsets of $\mathbb{C}\backslash[0, \infty)$. We have

$$Y_{n+1}(x) = 1 + l(1+\mu)\, Y_n(x) - l(1+\mu)\, Y_n(x) - (1+\mu)\, \frac{k_n^{(\alpha)}}{\tilde{k}_n}\, (n+1) \times \frac{L_n^{(\alpha-1)}(x)}{L_{n+1}^{(\alpha-1)}(x)}\, Y_n(x). \tag{20}$$

Denote

$$\delta_n(x) := -\left(l(1+\mu) + (1+\mu)\, \frac{k_n^{(\alpha)}}{\tilde{k}_n}\, (n+1)\, \frac{L_n^{(\alpha-1)}(x)}{L_{n+1}^{(\alpha-1)}(x)} \right) Y_n(x),$$

then

$$\lim_{n \to \infty} \delta_n(x) = 0, \tag{21}$$

locally uniformly on $\mathbb{C}\backslash[0, \infty)$.

Let us define $g_n(x) = Y_n(x) - 1/(1 - l(1+\mu))$. Then, (20) can be rewritten as

$$g_{n+1}(x) = l(1+\mu)\, g_n(x) + \delta_n(x)$$

and from (19) and (21) the fact that $g_n(x) \to 0$ is straightforward. ∎

Corollary 1 is an immediate consequence of Perron's formula and Theorem 2.

Proof of Corollary 2. For $\alpha > 0$ it is a trivial consequence of Perron's formula, Theorem 2 and the fact that $L_n^{(\alpha-1)}$ has all its zeros in $[0, \infty)$. For $\alpha = 0$, the zeros of $L_n^{(-1)}$ are those of $L_{n-1}^{(1)}$ (see [13, p. 102]) and 0 with multiplicity one. Then, again, the result is valid. When $-1 < \alpha < 0$, $L_n^{(\alpha-1)}$ has $n-1$ positive zeros and a negative zero (see [13, p. 151]). Again, using Perron's formula the negative zero goes to 0 when $n \to \infty$, and therefore the result holds in this situation. ∎

Proof of Theorem 3. From Fejér's formula for Laguerre polynomials (see [13, Theor. 8.22.1, p. 198]) we have

$$\frac{L_n^{(\alpha-1)}(x)}{n!\,n^{\alpha/2-3/4}} = (-1)^n \pi^{-1/2} e^{x/2} x^{-\alpha/2+1/4} \cos\left(2\sqrt{nx} - (\alpha-1)\frac{\pi}{2} - \frac{\pi}{4}\right) + O(n^{-1/2}). \tag{22}$$

Now, if we use the asymptotic formula (see [13, (1.71.7), p. 15]),

$$J_\alpha(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2}), \qquad z \to \infty,$$

with $z = 2\sqrt{nx}$, $n \to \infty$, then we get

$$\frac{(-1)^n L_n^{(\alpha-1)}(x)}{n!\,n^{\alpha/2-1/2}} = e^{x/2} x^{-\alpha/2+1/2} J_{\alpha-1}(2\sqrt{nx}) + O(n^{-3/4}). \tag{23}$$

We denote $a_n = -(1+\mu)(n+1)\,k_n^{(\alpha)}/\tilde{k}_n$ and using (15) we have

$$\begin{aligned}
&\frac{Q_{n+1}(x)}{(n+1)!\,(n+1)^{\alpha/2-1/2}}\\
&\quad= \frac{L_{n+1}^{(\alpha-1)}(x)}{(n+1)!\,(n+1)^{\alpha/2-1/2}} + \frac{1}{n^{1/4}}\frac{a_n}{n+1}\left(\frac{n}{n+1}\right)^{\alpha/2-1/2} \frac{Q_n(x)}{n!\,n^{\alpha/2-3/4}}\\
&\quad= \frac{L_{n+1}^{(\alpha-1)}(x)}{(n+1)!\,(n+1)^{\alpha/2-1/2}} + \frac{1}{n^{1/4}}\frac{a_n}{n+1}\left(\frac{n}{n+1}\right)^{\alpha/2-1/2}\\
&\qquad\times\left(\frac{L_n^{(\alpha-1)}(x)}{n!\,n^{\alpha/2-3/4}} + \frac{a_{n-1}}{n}\left(\frac{n-1}{n}\right)^{\alpha/2-3/4} \frac{Q_{n-1}(x)}{(n-1)!\,(n-1)^{\alpha/2-3/4}}\right).
\end{aligned} \tag{24}$$

If we define $R_n(x) := Q_n(x)/(n!\,n^{\alpha/2-3/4})$, (24) can be rewritten as

$$\frac{Q_{n+1}(x)}{(n+1)!\,(n+1)^{\alpha/2-1/2}} = \frac{L_{n+1}^{(\alpha-1)}(x)}{(n+1)!\,(n+1)^{\alpha/2-1/2}} + \frac{1}{n^{1/4}} \times \frac{a_n}{n+1}\left(\frac{n}{n+1}\right)^{\alpha/2-1/2} R_n(x), \tag{25}$$

where

$$R_n(x) = \frac{L_n^{(\alpha-1)}(x)}{n!\,n^{\alpha/2-3/4}} + \frac{a_{n-1}}{n}\left(\frac{n-1}{n}\right)^{\alpha/2-3/4} R_{n-1}(x).$$

On the other hand, for n large enough and x on compact subsets of $(0, \infty)$ by using (22) $L_n^{(\alpha-1)}(x)/(n!\, n^{\alpha/2-3/4})$ is uniformly bounded and by Proposition 2

$$\lim_{n\to\infty} \frac{a_{n-1}}{n} = -(1+\mu)\, l \qquad \text{with} \quad |(1+\mu)\, l| < 1.$$

Thus, we can conclude that $R_n(x)$ is uniformly bounded on compact subsets of $(0, \infty)$. Therefore, from (25) we get

$$\frac{Q_{n+1}(x)}{(n+1)!\,(n+1)^{\alpha/2-1/2}} = \frac{L_{n+1}^{(\alpha-1)}(x)}{(n+1)!\,(n+1)^{\alpha/2-1/2}} + O(n^{-1/4}), \qquad n\to\infty,$$

and by using (23) we obtain the result. ∎

Proof of Proposition 3. The inequality on the right hand side is a straightforward consequence of Proposition 1. On the other hand, from the extremal property of $k_n^{(\alpha)}$, i.e.,

$$k_n^{(\alpha)} = \inf\{\langle p, p\rangle : p(x) = x^n + \text{terms of lower degree}\},$$

we get

$$\begin{aligned}\tilde{k}_n = (Q_n, Q_n)_S &= \langle Q_n + \mu Q_n', Q_n + \mu Q_n'\rangle + (\lambda - \mu^2)\langle Q_n', Q_n'\rangle \\ &\geqslant k_n^{(\alpha)} + (\lambda-\mu^2)\, n^2 k_{n-1}^{(\alpha)}. \quad ∎\end{aligned}$$

Proof of Theorem 4. Denote by $l_n^{(\alpha)}$ the nth orthonormal Laguerre polynomial. From Theorem 1(d) and (e) we have

$$x l_n^{(\alpha)}(x) = a_{n+1} l_{n+1}^{(\alpha)}(x) + b_n l_n^{(\alpha)}(x) + a_n l_{n-1}^{(\alpha)}(x), \qquad n \geqslant 1,$$

where $a_n = \sqrt{n(n+\alpha)}$ and $b_n = 2n+\alpha+1$.

Since, for $j \in \mathbb{R}$ fixed,

$$\lim_{n\to\infty} \frac{a_n}{n+j} = 1, \qquad \lim_{n\to\infty} \frac{b_n}{n+j} = 2,$$

we will use the following result due to W. Van Assche (see [14, p. 117] or [15, p. 435]),

$$\lim_{n\to\infty} \frac{l_{n-1}^{(\alpha)}((n+j)\,x)}{l_n^{(\alpha)}((n+j)\,x)} = \frac{1}{\varphi\left(\dfrac{x-2}{2}\right)}, \qquad j \in \mathbb{R} \qquad \text{fixed}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$. Then

$$\lim_{n\to\infty} n\,\frac{L_{n-1}^{(\alpha-1)}((n+j)\,x)}{L_n^{(\alpha-1)}((n+j)\,x)}=\frac{1}{\varphi\left(\dfrac{x-2}{2}\right)},\qquad j\in\mathbb{R}\qquad \text{fixed} \tag{26}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$.

Now, if we make the change of variable $x\to nx$ in (a) of Corollary 5 and dividing by $L_n^{(\alpha-1)}(nx)$, we get

$$\frac{Q_n(nx)}{L_n^{(\alpha-1)}(nx)}=\sum_{j=0}^{n}(-1)^j\,b_j^{(n)}\,\frac{L_{n-j}^{(\alpha-1)}(nx)}{L_n^{(\alpha-1)}(nx)}. \tag{27}$$

First, we analyze the asymptotic behaviour of the coefficient $b_j^{(n)}$, for j fixed and $n\to\infty$. It is easy to observe from (16) that

$$\lim_{n\to\infty}\frac{b_j^{(n)}}{n(n-1)\cdots(n-j+1)}=(1+\mu)^j\lim_{n\to\infty}\prod_{i=1}^{j}\frac{k_{n-i}^{(\alpha)}}{\tilde{k}_{n-i}}=((1+\mu)\,l)^j, \tag{28}$$

and, using (26), we get for j fixed,

$$\lim_{n\to\infty}\prod_{i=1}^{j}(n-i+1)\,\frac{L_{n-j}^{(\alpha-1)}(nx)}{L_n^{(\alpha-1)}(nx)}=\varphi\left(\frac{x-2}{2}\right)^{-j}, \tag{29}$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$. Notice that $|\varphi((x-2)/2)|>1$ when $x\in\mathbb{C}\setminus[0,4]$, hence $|\varphi((x-2)/2)|^{-j}\leqslant 1$. Actually, if $j\geqslant 1$ the above inequality is strictly less than one and, if j is large, then it is $\ll 1$.

We denote

$$g_{n,j}(nx)=\begin{cases}(-1)^j\,b_{n-j}^{(n)}\,\dfrac{L_{n-j}^{(\alpha-1)}(nx)}{L_n^{(\alpha-1)}(nx)}, & 0\leqslant j\leqslant n,\\[2ex] 0, & j>n,\end{cases}$$

then, from (28) and (29), for j fixed we have

$$\lim_{n\to\infty} g_{n,j}(nx)=\left(\frac{-(1+\mu)\,l}{\varphi\left(\dfrac{x-2}{2}\right)}\right)^j:=g_j(x), \tag{30}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0, 4]$, and since $|(1+\mu)\, l|<1$ we get

$$g_j(x)=\left|\frac{l(1+\mu)}{\varphi\left(\frac{x-2}{2}\right)}\right|^j<1.$$

Moreover, if x belongs to a compact subset of $\mathbb{C}\backslash[0, 4]$, using (7), (29) and the fact that $\mu\in(-1, 0]$ (this guarantees that $(1+\mu)\, k_n^{(\alpha)}/\tilde{k}_n\leqslant\mathscr{C}$, for all $n\in\mathbb{N}$), we have for n large enough and $0\leqslant j\leqslant n$,

$$|g_{n,\, j}(nx)|\leqslant\mathscr{M}\mathscr{C}^j, \tag{31}$$

where $\mathscr{M}$ is a constant and $\mathscr{C}$ is the constant given by (8).

We have

$$\frac{Q_n(nx)}{L_n^{(\alpha-1)}(nx)}=\sum_{j=0}^{n} g_{n,\, j}(nx). \tag{32}$$

Then, by (31), we have a dominant for (32) and so the validity of the interchange of limit and summation is guaranteed by Lebesgue's dominated convergence theorem. Therefore, using (30), we get

$$\lim_{n\to\infty}\frac{Q_n(nx)}{L_n^{(\alpha-1)}(nx)}=\sum_{j=0}^{\infty} g_j(x)=\frac{\varphi\left(\frac{x-2}{2}\right)}{\varphi\left(\frac{x-2}{2}\right)+(1+\mu)\, l}.$$

Here, we can give information about the asymptotic behavior of $\{Q_n(nx)/L_n^{(\alpha)}(nx)\}$. Using Theorem 1(a) and Lemma 1 we can write

$$L_n^{(\alpha)}(nx)+nL_{n-1}^{(\alpha)}(nx)=Q_n(nx)+(1+\mu)\, n\,\frac{k_{n-1}^{(\alpha)}}{\tilde{k}_{n-1}}\, Q_{n-1}(nx),$$

then, if we divide by $L_n^{(\alpha)}(nx)$ it is possible, with minor changes, to proceed as above in order to prove (10). ∎

Proof of Corollary 3. This is a straightforward consequence of Theorem 4 and the fact that the contracted zeros of Laguerre polynomials cluster on the interval $[0, 4]$ (see [14] or [15]) and their asymptotic distribution has density $(2\pi)^{-1}\sqrt{4-x}/\sqrt{x}$ (see, for example, [14, p. 123]). ∎

Proof of Corollary 4. Since the sequence $\{c_n=n, n=1, 2, ...\}$ is a regularly varying sequence with index one (see [14, p. 120] or [15, p. 435]), we use a result due to J. S. Geronimo and W. Van Assche [3, for (4.2)] about the

Plancherel–Rotach asymptotics for Laguerre polynomials, that is, for monic Laguerre polynomials we get,

$$\lim_{n\to\infty} \frac{2^n \sqrt{2\pi n}\, L_n^{(\alpha)}(nx)}{n!\,(x-2+\sqrt{x^2-4x})^n \exp\left(\dfrac{2nx}{x+\sqrt{x^2-4x}}\right)}$$
$$= 2^{-\alpha-1/2} x^{-\alpha}(x-2+\sqrt{x^2-4x})^{1/2}\,(x+\sqrt{x^2-4x})^{\alpha}\,(\sqrt{x^2-4x})^{-1/2}, \tag{33}$$

taking into account that the square roots in the above formula are negative if x is negative. Therefore, by using (33) and Theorem 4, we obtain the Plancherel–Rotach asymptotics for Sobolev polynomials $Q_n(nx)$. ∎

Proof of Theorem 5. We can find a Mehler–Heine type formula for Laguerre polynomials in (13, p. 193]. If $\alpha \in \mathbb{R}$, then

$$\lim_{n\to\infty} \frac{(-1)^n L_n^{(\alpha)}(x/n)}{n^{\alpha} n!} = x^{-\alpha/2} J_{\alpha}(2\sqrt{x}), \tag{34}$$

uniformly on compact subsets of $\mathbb{C}$. Indeed, the proof of (34) is also valid in the following situation: for a fixed $j \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{(-1)^n L_n^{(\alpha)}(x/(n+j))}{(n+j)^{\alpha}\, n!} = x^{-\alpha/2} J_{\alpha}(2\sqrt{x}), \tag{35}$$

uniformly on compact subsets of $\mathbb{C}$.

Since,

$$\frac{(-1)^n Q_n(x/n)}{n^{\alpha-1} n!} = \sum_{j=0}^{n} \frac{b_j^{(n)}}{\prod_{i=0}^{j-1}(n-i)} \frac{(-1)^{n-j} L_{n-j}^{(\alpha-1)}(x/n)}{n^{\alpha-1}(n-j)!}, \tag{36}$$

with the assumption $b_0^{(n)} = 1$ and the convention that $\prod_{i=0}^{-1}(n-i) = 1$.

Therefore, if we take x on a compact subset of $\mathbb{C}$, then using (7), (35) and also the fact that $\mu \in (-1, 0]$, we again have a dominant for (36) and we can proceed as in Theorem 4 to prove (11). ∎

Proof of Proposition 4. This is a straightforward consequence of Theorem 5. ∎

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